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Reg. No.................................

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M.Sc. DEGREE (C.S.S.) EXAMINATION, MAY 2018

Fourth Semester

Faculty of Science

Branch I (A) : Mathematics

MT 04 C16—SPECTRAL THEORY

(Programme–Core–Common for all)

(2012 Admission onwards)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Suppose (x_n) is a sequence in a normed space X such that $x_n \xrightarrow{w} x$. Show that the weak limit x of (x_n) is unique.

2. Suppose $T:[1,\infty)\to[1,\infty)$ defined by $Tx = x + \frac{1}{x}$. Show that $|Tx - Ty| < |x - y|$ when $x \neq y$, but the mapping has no fixed point.

3. Find eigenvalues and eigenvectors of $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

4. Let X be a complex Banach space. Let S and $T \in B(X, X)$. Then show that $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda, \lambda, \mu \in \rho(T)$.

5. Define compact linear operator. Let X and Y be normed spaces. Prove that every compact linear operator $T : X \to Y$ is bounded.

6. Let X and Y be normed spaces and let $T : X \to Y$ be a compact linear operator. Prove that the range $\mathcal{R}(T)$ is separable.

Turn over

7. Let $T : H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space H. Then prove that all the eigen values of T (if they exists) are real.

8. Let P_1 and P_2 be projections of a Hilbert space H onto Y_1 and Y_2, respectively, and $P_1P_2 = P_2P_1$. Show that $P_1 + P_2 - P_1P_2$ is a projection of H onto $Y_1 + Y2$.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Let X be a normed space. Prove that $x_n \xrightarrow{w} x$ if and only if :

   (i) The sequence $\| (x_n) \|$ is bounded.

   (ii) For every element f of a total subset $M \subset X'$ we have $f(x_n) \to f(x)$.

10. Let X and Y be two normed spaces. On the product space $X \times Y$ define :

    (a) $\| (x, y) \| = \| x \| + \| y \|$ and

    (b) $\| (x, y) \| = \max \{\| x \|, \| y \|\}$

    Verify that (i) and (ii) are norms in $X \times Y$.

11. State and prove Banach fixed point theorem.

12. Prove that all matrices representing a given linear operator $T : X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigen values.

13. Prove that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space is closed.

14. Show that for any operator $T \in B(X, X)$ on a complex Banach space X, $r_\sigma(\alpha T) = \alpha r_\sigma(T)$, and $r_\sigma(T^k) = [r_\sigma(T)]^k$ $(k \in \mathbb{N})$ where r_σ denotes the spectral radius.

15. Let T be linear operator defined on all of a complex Hilbert space H and satisfies $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$. Prove that T is bounded.

16. Prove that a bounded linear operator $P : H \to H$ on a Hilbert space H is a projection if and only if P is self adjoint and idempotent.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. State and prove open mapping theorem.
18. State and prove spectral mapping theorem for polynomials.
19. Let B be a subset of metric space X. Then prove that :
    (i) If B is relatively compact, B is totally bounded.
    (ii) If B is totally bounded and X is complete, B is relatively compact.
    (iii) If B is totally bounded, for every $c > 0$ it has finite c-net. $M_c \subset B$.
    (iv) If B is totally bounded, B is separable.
20. (i) Prove that the set of eigenvalues of a compact linear operator $T : X \to X$ on a normed space X is countable and the only possible point of acccumulation is $\lambda = 0$.
    (ii) Let $T : X \to X$ be a compact linear operator and $S : X \to X$ a bounded linear operator on a normed space. Then prove that ST and TS are compact.
21. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Prove that a number λ belongs to the resolvent set $\rho(T)$ of T if and only if there exists a $c > 0$ such that for every $x \in H$, $\| T_\lambda x \| \geq c \| x \|$, $(T_\lambda = T - \lambda I)$.
22. Let H be a Hilbert space. Let P_1, P_2 be projections on H. Prove that :
    (i) $P = P_1 P_2$ is a projection H if and only if the projections P_1 and P_2 commutes. Then P projects H onto $Y = Y_1 \cap Y_2$, where $Y_i = P_i(H)$, $i = 1, 2$.
    (ii) Two closed subspaces Y and V of H are orthogonal if and only if the corresponding projections satisfy $P_Y P_V = 0$.
    (iii) The sum $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

(3 × 5 = 15)