F 5065 (Pages : 2) Reg. No..............................

Name..............................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2016

Third Semester

Faculty of Science

Branch : I (A) Mathematics

MT 03 C12—FUNCTIONAL ANALYSIS

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Give an example of a subspace of l^∞ which is not closed.
2. Show by an example that a closed and bounded subset of a metric space need not be compact.
3. Let X be a finite dimensional vector space show that if $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$, then $x_0 = 0$.
4. Show that the space l^p with $p \neq 2$ is not a Hilbert space.
5. Define a sesquilinear form on $X \times Y$, where X and Y are vector spaces.
6. Let $T: H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces, show that $(T^*)^* = T$.
7. State Baire's category theorem.
8. Show that l^1 is not reflexive.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Prove that every finite dimensional normed space is complete.
10. State and prove Riesz's lemma.
11. Define equivalent norms. Show that on a finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.

Turn over

12. Prove that in an inner product space X, the norm corresponding to the inner product satisfies $\|x+y\| \leq \|x\| + \|y\|$, where the equality sign holds if and only if $y^2 = 0$ or $x = cy, c \geq 0$.

13. State and prove Riesz representation theorem on Hilbert spaces.

14. Prove that the dual space X' of a normed space X is separable, then X itself is separable.

15. Prove that every Hilbert space is reflexive.

16. State Zorn's lemma. Prove that every vector space $X \neq [0]$ has a Hamel basis.

$(5 \times 2 = 10)$

Part C

Answer any **three** *questions.*

Each question has weight 5.

17. Let T be a linear operator. Then prove that (a) The range $\mathcal{R}$ (T) is a vector space ; (b) If dim $\mathcal{D}$ (T) $n < \infty$, then dim $\mathcal{R}$ (T) $\leq n$; and (c) The null space $\mathcal{N}$ (T) is a vector space.

18. If Y is a Banach space, then prove that B (X, Y) is a Banach space.

19. Show that the dual of l^p is l^q, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

20. (a) Let Y be any closed subspace of a Hilbert space H, show that $H = Y \oplus Z$, where $Z = Y^{\perp}$.

    (b) Prove that for any subset $M \neq \phi$ of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$.

21. Prove that two Hilbert spaces H and $\tilde{H}$, both real or both complex, are isomorphic if and only if they have the same Hilbert dimension.

22. State and prove the generalized Hahn-Banach theorem.

$(3 \times 5 = 15)$