F 4675 (Pages : 3) Reg. No.................................

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M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2017

Third Semester

Faculty of Science

Branch : I (A)—Mathematics

MTO 3C 13—DIFFERENTIAL GEOMETRY

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *out of eight questions.*
Each question has weight 1.

1. Define an integral curve of a vector field. What do you mean by a maximal integral curve of a vector field ?

2. Show by an example that the set of vectors tangent at a point p of a level set need not in general be a vector subspace of R_p^{n+1}.

3. Define a geodesic and show that geodesics have constant speed.

4. Define a Levi-Civita parallel vector field. Also show that if X and Y are two Levi-Civita parallel vector fields along α, $X \cdot Y$ is constant along α.

5. Define global parametrization of a plane curve.

6. Define a differential 1-form on an open set $U \subset R^{n+1}$. Define the 1-form dual of a vector field X on U.

7. Define the second fundamental form of a surface S. Also define the Gauss-Kronecker curvature of a surface at a point p.

8. Let ϕ be the map from the open square $0 < \theta < 2\pi, 0 < \phi < 2\pi$ into R^3 defined for $a > b > 0$ by :

   $\phi(\theta, \phi)((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$. What is ϕ^{-1} ?

(5 × 1 = 5)

Turn over

Part B

Answer any **five** *questions out of eight questions.*
Each question has weight 2.

9. Find the integral curve through $p=(1,1)$ of the vector field given by $X(x_1, x_2)=(x_2, x_1)$.

10. Let $S \subset R^{n+1}$ be a connected n-surface in R^{n+1}, show that there exist on S exactly two smooth unit normal vector fields N_1 and N_2 and $N_2(p) = -N_1(p)$ for all $p \in S$.

11. Describe the spherical image when $n = 1$ and when $n = 2$ of the n-surface oriented by $\nabla f / \| \nabla f \|$, where f is the function defined on the left side of the equation $-x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 0, x_1 > 0$.

12. Let S be an n-surface in R^{n+1}, $p, q \in S$ and let α be a piecewise smooth parametrized curve from p to q. Prove that the parallel transport $p_2 : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

13. Compute $\nabla v\, f$, where $f : R^{n+1} \to R, V \in R_p^{n+1}, p \in R^{n+1}$ given by :

$$f(x_1, x_2) = x_1^2 - x_2^2, v = (1, 1, \cos\theta, \sin\theta), n = 1.$$

14. Find the global parametrization of the plane curve oriented by $\nabla f / \| \nabla f \|$, where f is the function defined by the left side of the equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \geq 1, a \neq 0, b \neq 0$.

15. Find the Gaussian curvature of the parametrized z-surface $\phi(t, \theta) = (\cos\theta, \sin\theta, t)$.

16. Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self-adjoint linear transformation on V. Show that there exists on orthonormal basis for V consisting of eigen vector of L.

(5 × 2 = 10)

Part C

Answer any **three** *out of six questions.*
Each question has weight 5.

17. State and prove the existence and uniqueness theorem for integral curves for smooth vector fields.

18. Let S be an n-surface in R^{n+1}, $\alpha : I \to S$ be a parametrized curve in S, $t_o \in I$ and $v \in S_{\alpha}(t_o)$. Prove that there exists a unique vector field V tangent to S along α, which is parallel and has $V(t_o) = v$.

19. (a) Prove that the Weingarten map L_p is self adjoint.

    (b) Prove that local parametrizations of plane curves are unique upto reparametrization.

20. Let C be an oriented plane curve prove that there exists a global parametrization of C, if and only if C is connected.

21. (a) Find the Gauss-Kronecker curvature of a cylinder over an n-surface.

    (b) State and prove inverse function theorem for n-surfaces.

22. Let S be an n-surface in R^{n+1} and let $f : S \to R^k$. Prove that f is smooth if and only if $f \circ \phi + U \to R^k$ is smooth for each local parametrization $\phi : U \to S$.

(3 × 5 = 15)