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Name..................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JUNE 2016

Fourth Semester

Faculty of Science

Branch I (A) : Mathematics

MT 04 C16—SPECTRAL THEORY

(Programme Core—Common for all)

[2012 Admissions—Regular]

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question carries weight 1.

1. Define weak convergence. Show that the weak limit of the sequence (x_n) is unique.

2. Show that uniform operator convergence $T_n \to T, T_n \in B(X, Y)$ implies strong operator convergence with the same limit.

3. Show that the spectrum of a self adjoint linear operator in a finite dimensional inner product space is real.

4. Show that the set of all linear operator on a vector space into itself forms an algebra.

5. Prove that : $T : l^2 \to l^2$ defined by $T(\zeta_j) = (\zeta_j / j)$ for $j = 1, 2...$ is compact.

6. Let X and Y be normed spaces. Show that every compact linear operators are continuous.

7. Show that the difference of two projections $P_2 - P_1$ is a projection on a Hilbert space H if and only if $P_1 \leq P_2$.

8. Show that the residual spectrum σ_r (T) of a bounded self-adjoint linear operator T on a complex Hilbert space is empty.

(5 × 1 = 5)

Turn over

Part B

Answer any **five** *questions.*
Each question carries weight 2.

9. Show that in a finite dimensional normed space, weak convergence of a sequence implies strong convergence.
10. State and prove closed graph theorem.
11. Let X and Y be normed spaces, $T \in B(X, Y)$ and (x_n) a sequence in X. If $x_n \xrightarrow{w} x_0$, show that $Tx_n \xrightarrow{w} Tx_0$.
12. Show that the spectrum of a bounded linear operator on a complex Banach space is compact.
13. Suppose that A is a complex Banach algebra with identity. Show that the set of all invertible elements of A is an open subset of A.
14. Find $\sigma(T)$ for the operator $T : C[0,1] \to C[0,1]$ defined by $Tx = vx$, where $v \in X$ is fixed.
15. Prove that a self-adjoint linear operator T defined on all of a complex Hilbert space is bounded.
16. Show that the residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H is empty.

$(5 \times 2 = 10)$

Part C

Answer any **three** *questions.*
Each question carries weight 5.

17. State and prove open mapping theorem.
18. Let T be a bounded linear operator on a complex Banach space. Show that the spectral radius :

$$r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}.$$

19. Let $T \in B(X, X)$, where X is a Banach space. If $\|T\| < 1$, then prove that $(1 - T)^{-1}$ exists as a bounded linear operator on the whole space X and $(1 - T)^{-1} = \sum_{j=0}^{\infty} T^j$.

20. Let $T: X \rightarrow X$ be a compact linear operator on a normed space X. Show that for every $\lambda \neq 0, R(T - \lambda I)$ is closed.

21. If two bounded self-adjoint linear operator S and T on a Hilbert space H are positive and commutes, then show that ST is positive.

22. For any bounded self-adjoint linear operator T on a complex Hilbert space H, prove that :

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

(3 × 5 = 15)