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Name......................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JUNE 2016

Fourth Semester

Faculty of Science

Branch I (A) : Mathematics

MT 04 E01—ANALYTIC NUMBER THEORY

(2012 Admissions—Regular)

Time : Three Hours Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Define Euler's totient. Discuss its properties.
2. Show that Dirichlet product of two multiplicative function is also multiplicative.
3. Define Liouville's function. Show that it is completely multiplicative.
4. Let $f(x) = x^2 + x + 41$. Find the smallest integer $x \geq 0$ for which $f(x)$ is composite.
5. State Abel's identity.
6. Show that $ax \equiv b \pmod{m}$ has exactly one solution if $(a, m) = 1$.
7. Let $(a, m) = 1$. Show that a is a primitive root mod m if and only if the numbers $a, a^2, \dots, a^{\phi(m)}$ form a reduced residue system mod m.
8. Prove that $5n^3 + 7n^5 \equiv \pmod{12}$ for all integers n.

$(5 \times 1 = 5)$

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. For $n \geq 1$, prove that $\log n = \sum_{d|n} \Lambda(d)$.
10. For $x \geq 1$, prove that $\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right)$.
11. State and prove Legendre's identity.

Turn over

12. Describe Chebyshev's functions $\psi(x)$ and $\vartheta(x)$.

13. State and prove Wilson's theorem.

14. State and prove Chinese Remainder theorem.

15. Let g be a primitive root mod p, where p is an odd prime. Prove that even powers $g^2, g^4, \dots, g^{p-1}$ are quadratic residues mod p and odd powers $g^1, g^3, \dots, g^{p-2}$ are quadratic non-residues mod p.

16. State and prove Wolstenhome's theorem.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. Derive Dirichlet's formula for the partial sums of the divisor function $d(n)$.

18. For $x \geq 1$, prove that $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O\left(x^{-s}\right)$ if $s > 0, s \neq 1$.

19. For every integer $n \geq 2$, prove that :

$$\frac{1}{6} \cdot \frac{n}{\log n} < \pi(n) < 6 \cdot \frac{n}{\log n}.$$

20. Prove that the set of lattice points in the plane visible from the origin contains arbitrarily large gaps.

21. State and prove Wolstenhome's theorem.

22. Let 'p' be an odd prime and let 'd' be any positive divisor of $p-1$. Show that every reduced residue system mod p has exactly $\phi(d)$ numbers a such that $\exp_p(a) = d$.

(3 × 5 = 15)