**F 7733** (Pages : 4) Reg. No.................................

Name....................................

# M.Sc. DEGREE (C.S.S.) EXAMINATION, FEBRUARY 2014

## First Semester

Faculty of Science

Branch I (A)—Mathematics

MTO IC 01—LINEAR ALGEBRA

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

### Part A

*Answer any* **five** *questions.*
*Each question has weight* 1.

1. Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$ and $\alpha_3 = (0, -3, 2)$ forms a basis for $R^3$.

2. Find three vectors in $R^3$ which are linearly dependent and are such that any *two* of them are linearly independent.

3. Let F be a field and let T be a linear operator on $F^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$. Find $T^{-1}$.

4. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $C^3$ defined by $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1)$ and $\alpha_3 = (2, 2, 0)$. Find the dual basis of B.

5. Let D be a 2-linear function with the property that $D(A) = 0$ for all $2 \times 2$ matrices A over K having equal rows. Show that D is alternating.

6. Let K be a commutative ring with identity and let $n$ be a positive integer. Show that there exists atleast one determinant function on $K^{n \times n}$.

7. Let V be an $n$-dimensional vector space over F. Find the characteristic polynomials of the identity operator and zero operator.

8. Find an invertible real matrix P such that $P^{-1}AP$ and $^{-1}BP$ are both diagonals, were

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}.$$

$(5 \times 1 = 5)$

Turn over

## Part B

*Answer any* **five** *questions.*
*Each question has weight* 2.

9. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $R^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \quad \frac{4}{3}x_3 - x_5 = 0.$$

$$9x_1 - \quad 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0.$$

Find a finite set of vectors which spans W.

10. Let T be a linear transformation from V into W. Show that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

11. Let $W_1$ and $W_2$ be subspaces of a finite dimensional vector space. Show that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

12. Let $C^{2\times 2}$ be the complex vector space of $2 \times 2$ matrices with complex entries. Let $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$ and let T be the linear operator on $C^{2\times 2}$ defined by T (A) = BA. What is the rank of T. Can you describe $T^2$.

13. Let K be a commutative ring with identify and let A and B $n \times n$ matrices. Prove that $\det(AB) = (\det A)(\det B)$.

14. Find a $3 \times 3$ matrix for which the minimal polynomial is $x^2$.

15. Let V be a finite dimensional vector space and let $W_1, W_2 \ldots W_n$ be subspaces of V such that $V = W_1 + \ldots + W_n$ and $\dim V = \dim W_1 + \ldots + \dim W_n$. Prove that $V = W_1 \oplus + \ldots \oplus W_n$.

16. Let T be a linear operator on an $n$ dimensional vector space V. Show that the characteristic and minimal polynomials for T have the same roots, except for multiplication.

$(5 \times 2 = 10)$

## Part C

*Answer any* **three** *questions.*
*Each question has weight* 5.

17. (a) Let R be a non-zero row reduced echelon matrix. Prove that the non-zero vectors of R form a basis for the row space of R.

    (b) Prove that the space of all $m \times n$ matrices over the field F has dimension $mn$, by exhibiting a basis for this space.

18. (a) Let V be a finite - dimensional vector space over the field F and let W be a subspace of V. Show that $\dim W + \dim W^0 = \dim V$.

    (b) Let $\alpha_1 = (1, 0, -1, 2)$ and $\alpha_2 = (2, 3, 1, 1)$ and let W be the subspace of $R^4$ spanned by $\alpha_1$ and $\alpha_2$, which linear functionals $f : f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator of W.

19. Let V be an $n$-dimensional vector space over the field F and let W be an $m$-dimensional vector space over F. Show that the space $L(V, W)$ is finite dimensional and has dimension $mn$.

20. (a) Let A be an $n \times n$ matrix over K. Prove that A is invertible over K if and only if det A is invertible over K, and $A^{-1} = (\det A)^{-1} \operatorname{adj} A$.

    (b) We Cramer's rule to solve the following system of linear equations over the field of rational numbers.

$$\begin{aligned} x + \ \ y + z &= 11 \\ 2x - 6y - z &= 0 \\ 3x + 4y + 2z &= 0. \end{aligned}$$

21. (a) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Show that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

    (b) If U is the linear operator on $C^2$, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$. Show that U has 1 dimensional invariant subspaces.

Turn over

22. (a) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p=(x-c_1)\dots(x-c_k)$, where $c_1, c_2 \dots c_k$ are distinct elements of F.

(b) Show that every matrix A such that $A^2 = A$ is similar to a diagonal matrix.

$(3 \times 5 = 15)$