**F 5110** (Pages : 3) Reg. No..............................

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# M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY/FEBRUARY 2017

**First Semester**

Faculty of Science

Branch I (a)—Mathematics

MT 01 C01—LINEAR ALGEBRA

(2012 Admission onwards)

Time : Three Hours Maximum Weight : 30

## Part A

*Answer any* **five** *questions.*
*Each question carries* 1 *weight.*

1. Define a vector space. Is R a vector space over C, the complex field.
2. Verify whether (3, –1, 0, –1) in the subspace of $R^5$ spanned by the vectors (2, – 1, 3, 2), (–1, 1, 1, –3) and (1, 1, 9, –5).
3. Describe the range and null space of the differentiation transformation.
4. Let T be the linear operator on $R^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$. What is the matrix of T in the standard ordered basis for $R^2$.
5. Prove that a linear combination of $n$-linear functions is $n$-linear.
6. If K is a commutative ring with identity and A is the matrix over K given by $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$. Show that det A = 0.
7. Prove that similar matrices have the same characteristic polynomial.
8. Find a 3 × 3 matrix for which the minimal polynomial is $x^2$.

(5 × 1 = 5)

## Part B

*Answer any* **five** *questions.*
*Each question carries* 2 *weight.*

9. Let V be the vector space of all 2 × 2 matrices over the field F. Let $W_1$ be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and $W_2$ be the set of matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$. Prove that $W_1$ and $W_2$ are subspaces of V and also find the dimension of $W_1 \cap W_2$.

Turn over

10. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional. Prove that rank (T) + nullity (T) = dim V.

11. Let V be a finite dimensional vector space over the field F, and let W be a subspace of V. Show that dim W + dim $W^{\circ}$ = dim V.

12. Let V be a finite-dimensional vector space over the field F. For each vector $\alpha$ in V define $L_{\alpha}(f) = f(\alpha), f \varepsilon V^{*}$. Show that the mapping $\alpha \rightarrow K_{\infty}$ is an isomorphism of V onto $V^{**}$.

13. Let $\alpha$ and $\tau$ be the permutations of degree 4 defined by $\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 4, \sigma_4 = 1, \tau_1 = 3, \tau_2 = 1, \tau_3 = 2, \tau_4 = 4$ :

    (a) Is $\sigma$ odd or even ? Is $\tau$ odd or even.

    (b) Find $\sigma\tau$ and $\tau\sigma$.

14. Let T be a linear operator on a finite-dimensional vector space V. Let $c_1, c_2, \ldots c_k$ be the distinct characteristic value of T and Let $W_i$ be the null space of $(T - c_i I)$. Show that the following are equivalent :

    (i) T is a diagonalizable.

    (ii) The characteristic polynomial for T is $f = (x - c_1)^{d_1} \ldots (x - c_k)^{dk}$ and dim $W_i = di$, $i = 1, 2, \ldots k$.

    (iii) dim $W_1 + \ldots +$ dim $W_k$ = dim V.

15. Find an invertible real matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, where A and B are given by $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix}$.

16. Let V be a finite-dimensional vector space and let $W_1, \ldots W_k$ be subspaces of V such that $V = W_1 + \ldots + W_k$ and dim V = dim $W_1 + \ldots +$ dim $W_k$. Prove that $V = W_1 \oplus \ldots \oplus W_k$.

(5 × 2 = 10)

## Part C

*Answer any* **three** *questions.*
*Each question carries* 5 *weight.*

17. Let W be the subspace of $C^3$ spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1 + i, 1, -1)$

    (a) Show that $\alpha_1$ and $\alpha_2$ form a basis for W.

    (b) Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in W and form another basis for W.

    (c) What are the co-ordinates of $\alpha_1$ and $\alpha_2$ in the ordered basis $\{\beta_1, \beta_2\}$ for W.

18. (a) Let V be an $n$-dimensional vector space over the field F and let W be an $m$-dimensional vector space over F. Prove that L (V, W) is finite dimensional and has dimension $mn$.

    (b) If $f$ is a non-zero linear functional on the vector space V. Prove that the null space of $f$ is a hyperspace in V. Also prove that every hyperspace in V is the null space of a non-zero linear functional on V.

19. (a) Let V be a finite dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \beta_2, \dots \beta_n$ be any vectors in W. Prove that there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$ for $j = 1, 2, \dots n$.

(b) Prove that every $n$-dimensional vector space over the field F is isomorphic to the space $F^n$.

20. Let A be an $n \times n$ matrix over $k$. Prove that A is invertible over $k$ if and only if det A is invertible in $k$. Also prove that when A is invertible $A^{-1} = (\det A)^{-1}$ adj A. Here $k$ is a commutative ring with identity.

21. State and prove Cayley-Hamilton theorem for linear operators.

22. (a) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomial over F.

(b) Find a projection F which projects $R^2$ onto the subspace spanned by $(1, -1)$ along the subspace spanned by $(1, 2)$.

$(3 \times 5 = 15)$