

B.Sc. DEGREE (C.B.C.S.S.) EXAMINATION, OCTOBER 2015**Fifth Semester****Core Course—ABSTRACT ALGEBRA**

(Common for Model I and Model II B.Sc. Mathematics)

[2013 Admissions]

Time : Three Hours

Maximum : 80 Marks

Part A

*Answer all questions.
Each question carries 1 mark.*

1. On Q^+ define $a * b = a/b$. Is $*$ a binary operation on Q^+ .
2. On Q^+ define $a * b = ab/2$. Find the inverse of a .
3. Find the order of the cyclic subgroup of Z_4 generated by 3.
4. Q under addition is a cyclic group. Write True or False.
5. Define a cyclic group.
6. Find the partition of Z_n into cosets of the subgroup $H = \{0, 3\}$.
7. Let $\phi: Z \rightarrow R$ under addition given by $\phi(n) = n$. Is ϕ a homomorphism.
8. What are the units of Z_{14} ?
9. G is an ideal in R . Write True or False.
10. Define a maximal ideal.

(10 × 1 = 10)

Part B

*Answer any eight questions.
Each question carries 2 marks.*

11. Prove that the identity element and inverse of each element are unique in a group.
12. Describe all the elements in the cyclic subgroup of $GL(2, R)$ generated by the 2×2 matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
13. Let G be a group and let $a \in G$. Show that $H_a = \{x \in G / xa = ax\}$ is a subgroup of G .
14. Find all orbits of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$.

Turn over

15. Prove that every cyclic group is abelian.
16. Exhibit the left cosets and the right cosets of the subgroup $3\mathbb{Z}$ of \mathbb{Z} .
17. Prove that a factor group of a cyclic group is cyclic.
18. Define a normal subgroup of a group.
19. Prove that a group homomorphism $\phi: G \rightarrow G'$ is a one-to-one map if and only if $\ker(\phi) = \{e\}$.
20. Solve the equation $x^2 - 5x + 6 = 0$ in \mathbb{Z}_{12} .
21. Show that the characteristic of a subdomain of an integral domain D is equal to the characteristic of D .
22. Give an example to show that a factor ring of an integral domain may be a field.

(8 × 2 = 16)

Part C

*Answer any six questions.
Each question carries 4 marks.*

23. Let A be a non-empty set and S_A be the collection of all permutations of A . Show that S_A is a group under permutation multiplication.
24. Let H be a subgroup of a group G . For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that \sim is an equivalence relation on G .
25. Show that a non-empty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.
26. Prove that a subgroup of a cyclic group is cyclic.
27. Let H be a normal subgroup of G . Show that the cosets of H form a group G/H under the binary operation $(aH)(bH) = (ab)H$.
28. Let S_n be the symmetric group of n letters and let $\phi: S_n \rightarrow \mathbb{Z}_2$ be defined by :

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is an even permutation} \\ 1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

show that ϕ is a homomorphism.

29. Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.
30. Let R be a ring with unity. If $n \cdot 1 \neq 0$ for all $n \in \mathbb{Z}^+$, show that R has characteristic 0. If $n \cdot 1 = 0$ for some $n \in \mathbb{Z}^+$, show that the smallest such n is the characteristic of R .
31. If R is a ring with unity and N is an ideal of R containing a unit, show that $N = R$.

(6 × 4 = 24)

Part D

*Answer any two questions.
Each question carries 15 marks.*

32. (a) Prove that every group is isomorphic to a group of permutations.
(b) Prove that no permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.
33. (a) State and prove fundamental theorem of homomorphism for groups.
(b) Prove that M is a maximal normal subgroup of G if and only if G/M is simple.
34. Let ϕ be a homomorphism of a group G into a group G' . Show that :
- (i) If e is the identity element in G , then $\phi(e)$ is the identity element e^1 in G' .
 - (ii) If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$.
 - (iii) If H is a subgroup of G , then $\phi(H)$ is a subgroup of G' .
 - (iv) If K' is a subgroup of G' , then $\phi^{-1}[K']$ is a subgroup of G .
35. (a) Let $\phi: R \rightarrow R'$ be a ring homomorphism with Kernel H . Prove that the additive cosets of H form a ring R/H whose binary operations are defined by choosing representatives. Also show that the map $\mu: R/H \rightarrow \phi[R]$ defined by $\mu(a + H) = \phi(a)$ is an isomorphism.
- (b) Let R be a commutative ring and let $a \in R$. Show that $I_a = \{x \in R | ax = 0\}$ is an ideal of R .

(2 × 15 = 30)