E 3210 (Pages : 4) Reg. No.............................

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B.Sc. DEGREE (C.B.C.S.S.) EXAMINATION, OCTOBER 2016

Fifth Semester

Core Course—ABSTRACT ALGEBRA

(Common for Model I and Model II B.Sc. Mathematics)

[2013 Admission onwards]

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. On Z^+ define $a * b = a \mid b$. Is * a binary operation on Z^+.
2. Define a group.
3. Is the group $< 6z, + >$ cyclic.
4. Every group of order ≤ 4 is cyclic.

Write True *or* False :

5. What is the symmetric group on n letters ? How many elements such a group has ?
6. A subgroup of a group is a left coset of itself. Write True *or* False.
7. Define Kernel of a homomorphism $\phi : G \to G^1$.
8. Find all the units of Z_5.
9. What is the characteristic of the ring Z_u ?
10. Define a prime ideal.

$(10 \times 1 = 10)$

Part B

Answer any **eight** *questions.*
Each question carries 2 *marks.*

11. Show that for all $a, b \in G$, where G is a group, $(a * b)^1 = b^1 * a^1$, a^1 denote the inverse of a.
12. Let * be defined on the set $\mathbb{R}^*$ of non-zero real numbers by letting $a * b = a \mid b$. Check whether * gives a group structure on $\mathbb{R}^*$.

Turn over

13. Show that if H and K are subgroups of an abelian group G, then $\{Gk \mid G \in H, k \in K\}$ is a subgroup of G.

14. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$. Compute $Z\sigma$.

15. Find all orders of subgroups of Z_8.

16. Prove that every group of prime order is cyclic.

17. Compute the factor group $(z_4 \times z_6) | < (0.2) >$.

18. Find the number of elements in the cyclic subgroup of Z_{30}, generated by 25.

19. Let ϕ be a homomorphism of a group G onto a group G^1. Show that if G is abelian, G^1 is also abelian.

20. Prove that every field is an integral domain.

21. Show that the characteristic of an integral domain D must be either O or a prime P.

22. Prove that a ring homomorphism $\phi : \mathbb{R} \to \mathbb{R}^1$ is a one-to-one map if and only if $\text{Ker}(\phi) = [0]$.

(8 × 2 = 16)

Part C

Answer any **six** *questions.*
Each question carries 4 marks.

23. Let G be a group and $a \in G$. Show that $H = \{a^n, n \in Z\}$ is a subgroup of G and is the smallest subgroup of G that contains a.

24. If G is a group with binary operation *, show that the left and right cancellation laws hold in G.

25. Prove that every cyclic group is abelian.

26. Show that intersection of normal subgroups of G is again a normal subgroup of G.

27. Prove that μ is a maximal normal subgroup of G if and only if G/μ is simple.

28. If $\phi : G \to G^1$ is a group homomorphism, show that $\text{Ker}(\phi)$ is a normal subgroup of G.

29. If $\mathbb{R}$ is a ring with identify. Show that for any $a, b \in \mathbb{R}$:

(i) $0a = a0 = 0$.

(ii) $a(-b) = (-a)b = -(ab)$.

(iii) $(-a)(-b) = ab$.

30. Let N be an ideal of a ring $\mathbb{R}$. Show that $r = \mathbb{R} \to \mathbb{R}|\mathrm{N}$ given by $r(x) = x + \mathrm{N}$ is a ring homomorphism with Kernel N.

31. Let $\mathbb{R}$ be a commutative ring and $a \in \mathbb{R}$. Show that $\mathrm{I}_a = \{x \in \mathrm{R} \mid ax = 0\}$ is an ideal of $\mathbb{R}$.

(6 × 4 = 24)

Part D

Answer any **two** *questions.*
Each question carries 15 *marks.*

32. (a) Prove that if $n \geq 2$, the collection of all even permutations of $\{1, 2, \ldots, n\}$ forms a subgroup of order $n\,!/2$ of the symmetric group S_n.

(b) Let G and G^1 be groups and let $\phi : G \to G^1$ be a one-to-one function such that $\phi(xy) = \phi(x)\,\phi(y)$ for all $x, y \in G$. Prove that $\phi[G]$ is a subgroup of G^1 and ϕ provides an isomorphism of G with $\phi[G]$.

33. (a) Let G be a cyclic group with generator a. Prove that if the order of G is infinite, then G is isomorphic to $< Z, + >$. Also show that if G has finite order n, then G is isomorphic to $< Z_n, +n >$.

(b) Prove that if it is a subgroup of a finite group G, then the order of H is a divisor of order of G.

34. (a) Show that every finite integral domain is a field.

(b) State and prove fundamental homomorphism theorem for groups.

35. (a) Let H be a subring of the ring $\mathbb{R}$. Show that multiplication of additive cosets of H is well defined by the equation $(a+H)(b+H) = ab + H$ if and only if $ah \in H$ and $hb \in H$ for all $a, b \in \mathbb{R}$ and $h \in H$.

(b) Show that if $\mathbb{R}$ is a ring with unity and N is an ideal of $\mathbb{R}$ such that $N \neq \mathbb{R}$, then $\mathbb{R}/N$ is a ring with unity.

(2 × 15 = 30)